

Bianchi I Quantum cosmology in the Bergmann-Wagoner theory

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Abstract

Dedicated to Heinz Dehnen in honour of his 65th birthday.

The Wheeler-DeWitt equation is considered in the context of generalized scalar-tensor theories of gravitation for the Bianchi type I cosmology. Exact solutions are found for two selfinteracting potentials and arbitrary coupling function. The WKB wavefunctions are obtained and a family of solutions satisfying the Hawking-Page regularity conditions of wormholes are found.

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1 Introduction

In this paper we consider quantum cosmological models for Bianchi type I models in the generalized vacuum scalar–tensor theories of gravity. Interest in these theories has been widespread in recent years in connection with inflation and string theories. They are defined by the action

$$A = \int d^4x \sqrt{-g} e^{-\Phi} \left[R - \omega(\Phi) (\nabla\Phi)^2 - 2\lambda(\Phi) \right], \quad (1)$$

where R is the Ricci curvature of the space–time and g is the determinant of the metric $g_{\mu\nu}$ [1]. The dilaton field Φ plays the role of a time–varying gravitational constant and may self–interact through a potential $\lambda(\Phi)$ (the usual Brans–Dicke scalar field is $\phi = \exp(-\Phi)$). The function $\omega(\Phi)$ plays the role of a coupling function between the dilaton and graviton. Each scalar–tensor theory is characterized by specific functional forms of $\omega(\Phi)$ and $\lambda(\Phi)$. A cosmological constant in the gravitational sector of the theory corresponds to the special case where $\lambda(\Phi)$ is a space–time constant. Among others we have the case of Brans–Dicke theory, where $\omega(\Phi)$ is a space–time constant and $\lambda(\Phi)$ is absent [2]. It is known that inflationary solutions exist in a wide class of scalar–tensor cosmologies [3], therefore these theories are relevant to the study of the very early Universe [4]. Also, higher–order [5] and higher–dimensional [6] theories of gravity may be expressed in a scalar–tensor form after suitable field redefinitions. Brans–Dicke theory with $\omega = -1$ corresponds to a truncated version of the string effective action [7].

Point symmetries associated with action (1) have been discussed previously within the context of the spatially isotropic Friedmann Universes [8, 9] and Bianchi models [10]. It was found that $\omega(\Phi)$ and $\lambda(\Phi)$ must be related in a certain way if the field equations are to be symmetric.

In this paper we consider the Wheeler–DeWitt equation (WDW) for the simplest anisotropic model, namely, the Bianchi type I cosmology. Some time ago the canonical formulation of the Brans–Dicke theory was considered by Toton and by Matzner et al. [11]. For a recent review of WDW in different theories of gravitation see Ref. [12] The line element for the class of spatially homogeneous space–times is given by

$$ds^2 = -dt^2 + h_{ab}\omega^a\omega^b, \quad a, b = 1, 2, 3, \quad (2)$$

where $h_{ab}(t)$ is a function of cosmic time t and represents the metric on the surfaces of homogeneity and ω^a are one–forms. These models have a topology $R \times G_3$, where

G_3 represents a Lie group of isometries that acts transitively on the space-like three-dimensional orbits [13]. The Lie algebra of G_3 admits the structure constants $C^a_{bc} = m^{ad}\epsilon_{dbc} + \delta^a_{[b}a_{c]}$, where m^{ab} is a symmetric matrix, $a_c \equiv C^a_{ac}$ and $\epsilon_{abc} = \epsilon_{[abc]}$. The Jacobi identity $C^a_{b[c}C^b_{de]} = 0$ is only satisfied if $m^{ab}a_b = 0$, so m^{ab} must be transverse to a_b [14]. The model belongs to the Bianchi class A if $a_b = 0$ and to the class B if $a_b \neq 0$. A basis may be found such that $a_b = (a, 0, 0)$ and $m^{ab} = \text{diag}[m_{11}, m_{22}, m_{33}]$, where m_{ii} take the values ± 1 or 0. In the Bianchi class A, the Lie algebra is uniquely determined up to isomorphisms by the rank and signature of m^{ab} . The six possibilities are $(0, 0, 0)$, $(1, 0, 0)$, $(1, -1, 0)$, $(1, 1, 0)$, $(1, 1, -1)$ and $(1, 1, 1)$ and these correspond, respectively, to the Bianchi types I, II, VI₀, VII₀, VIII and IX. Finally, the three-metric may be parametrized by $h_{ab}(t) = e^{2\alpha(t)} \left(e^{2\beta(t)} \right)_{ab}$, where $e^{3\alpha}$ represents the effective spatial volume of the Universe and

$$\beta_{ab} \equiv \text{diag} [\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+] \quad (3)$$

is a traceless matrix that determines the anisotropy in the models.

The configuration space Q for the Bianchi models derived from action (1) is therefore four-dimensional and is spanned by $\{q_n \equiv \alpha, \Phi, \beta_{\pm}\}$. The Lagrangian density $L(q_n, \dot{q}_n)$ is defined by $A = \int dt L(q_n, \dot{q}_n)$, where a dot denotes differentiation with respect to cosmic time. It may be derived by substituting the metric (2) into the action (1) and integrating over the spatial variables. This procedure is unambiguous for the class A cosmologies and the action for these models simplifies to [10]

$$A = \int dt e^{3\alpha - \Phi} \left[6\dot{\alpha}\dot{\Phi} - 6\dot{\alpha}^2 + 6\dot{\beta}_+^2 + 6\dot{\beta}_-^2 + \omega(\Phi)\dot{\Phi}^2 - 2\lambda(\Phi) + e^{-2\alpha}U(\beta_{\pm}) \right], \quad (4)$$

where

$$U(\beta_{\pm}) = -e^{-4\alpha} \left(m_{ab}m^{ab} - \frac{1}{2}m^2 \right) \quad (5)$$

is the curvature potential, $m \equiv m^a_a$ and indices are raised and lowered with h^{ab} and h_{ab} , respectively [15]. In the case of the type B models, a divergence may arise because the three-curvature contains a term proportional to $a_b a^b$ [16]. In view of this difficulty, we do not consider these models further.

To proceed we take the following changes of variables,

$$dt = e^{\Phi/2} d\tau, \quad x = \alpha - \Phi/2, \quad y = \int \sqrt{\frac{3 + 2\omega(\Phi)}{12}} d\Phi, \quad (6)$$

and the action is now

$$A = \int d\tau \left[6e^{3x} \left\{ y'^2 - x'^2 + \beta_+'^2 + \beta_-'^2 - \Lambda(y) \right\} + e^x U(\beta_{\pm}) \right], \quad (7)$$

where

$$\Lambda(y) := e^{\Phi} \lambda(\Phi)/3. \quad (8)$$

The above action means that the Lagrangian is

$$L = \left[6e^{3x} \left\{ y'^2 - x'^2 + \beta_+'^2 + \beta_-'^2 - \Lambda(y) \right\} + e^x U(\beta_{\pm}) \right]. \quad (9)$$

From here we can calculate the canonical momenta and the Hamiltonian

$$\pi_x = -12e^{3x} x', \quad \pi_y = 12e^{3x} y', \quad \pi_+ = 12e^{3x} \beta_+', \quad \pi_- = 12e^{3x} \beta_-', \quad (10)$$

$$H = \frac{e^{-3x}}{24} [-\pi_x^2 + \pi_y^2 + \pi_+^2 + \pi_-^2 + 144e^{6x} \Lambda(y) - 24e^{4x} U(\beta_{\pm})]. \quad (11)$$

Then the Wheeler-DeWitt equation follows from the canonical quantization of $H = 0$, i.e., the canonical momenta in Eq.(10) are converted into operators in the standard way, in general, owing to the ordering ambiguity we have

$$\pi_x^2 \rightarrow -\frac{1}{x^B} \frac{\partial}{\partial x} \left(x^B \frac{\partial}{\partial x} \right), \quad (12)$$

therefore the WDW equation $H\Psi(x, y) = 0$ for an arbitrary factor ordering, encoded in the B parameter, is

$$\left[\partial_x^2 - B\partial_x - \partial_y^2 - \partial_+^2 - \partial_-^2 - 24e^{4x} U(\beta_{\pm}) + 144e^{6x} \Lambda(y) \right] \Psi(x, y, \beta_{\pm}) = 0. \quad (13)$$

In the following section we try to solve the Wheeler-DeWitt equation by separation of variables in the simplest of the homogeneous cosmologies, namely the Bianchi Type I case in which the potential $U(\beta_{\pm})$ vanishes.

2 Exact solutions for Bianchi I

We now consider the case of Bianchi type I cosmological model for which the potential $U(\beta_{\pm})$ vanishes identically. Furthermore we restrict ourselves to the case when $\Lambda(y) = \Lambda_o/144 = \text{constant}$, i.e. $\lambda(\phi) \propto \phi$; this form of the potential has been used previously

to obtain classical solutions in Bianchi type I vacuum cosmology [17] and for isotropic models with a barotropic fluid[18, 19]; this potential is one of those that could produce inflation[3]. The choices that we have made simplify the WDW equation and allows us to obtain exact solutions by means of separation of variables,

$$\Psi(x, y, \beta_{\pm}) = X(x)Y(y)F(\beta_+)G(\beta_-). \quad (14)$$

This implies that equation(13)gives the separated equations

$$X'' - BX' + [\kappa_0^2 + \Lambda_0 e^{6x}]X = 0, \quad (15)$$

$$Y'' + k^2 Y = 0, \quad (16)$$

$$F'' + k_+^2 F = 0, \quad (17)$$

$$G'' + k_-^2 G = 0, \quad (18)$$

where

$$\kappa_0^2 = k_+^2 + k_-^2 + k^2. \quad (19)$$

The solutions to these equations give the wavefunction:

$$\begin{aligned} \Psi(x, y, \beta_{\pm}) = & e^{Bx/2} [c_1 J_{\nu}(\frac{\sqrt{\Lambda_0}}{3} e^{3x}) + c_2 Y_{\nu}(\frac{\sqrt{\Lambda_0}}{3} e^{3x})] [c_3 e^{iky} + c_4 e^{-iky}] \\ & [c_5 e^{ik_+\beta_+} + c_6 e^{-ik_+\beta_+}] [c_7 e^{ik_-\beta_-} + c_8 e^{-ik_-\beta_-}], \end{aligned} \quad (20)$$

where

$$\nu = \frac{\sqrt{-B - 4\kappa_0^2}}{6}, \quad (21)$$

and the c_i are constants. By superposition of these solutions, wavefunctions satisfying different boundary conditions can be obtained. In the following subsection we consider the case of wormholes.

2.1 Wormhole solution

Quantum wormholes can be regarded as a special class of solutions to the Wheeler-DeWitt equation with certain boundary conditions[20]: *i*) the wavefunction is exponentially damped for large spatial geometry, i.e., when $\alpha \rightarrow \infty$, *ii*) the wavefunction is regular when the spatial geometry degenerates, i.e., the wavefunction does not oscillate when $\alpha \rightarrow -\infty$. These conditions are known as the Hawking-Page regularity conditions (HP). In what follows we show, with a particular factor ordering, the explicit form of a wavefunction that satisfies the HP regularity conditions.

The following wavefunction can be obtained by superposition of the solutions of the previous section or it can be substituted into the WDW equations to check that it is a particular solution

$$\Psi(x, y, \beta_{\pm}) = e^{m \cosh[ny + p\beta_+ + q\beta_- + r] \exp(3x)}, \quad (22)$$

where r is an arbitrary real parameter, the factor ordering B and the constants m, n, p, q satisfy the relations

$$B = 3 + \Lambda_0/3, n^2 + p^2 + q^2 = 9, -(n^2 + p^2 + q^2)m^2 = \Lambda_0. \quad (23)$$

From these relations we see that for a wormhole we require a negative Λ_0 . If from the last relation we take the negative root for m , it is easy to check that the wavefunction is exponentially damped for large spatial geometry, i.e., when $\alpha \rightarrow \infty$ ($x \rightarrow \infty$) and also that the wavefunction does not oscillate when $\alpha \rightarrow -\infty$ ($x \rightarrow -\infty$).

2.2 WKB solution

Here we want to obtain the WKB wavefunction in the form

$$\Psi(\eta, \xi, \beta_{\pm}) = e^{i[S(x, y, \beta_+, \beta_-)]}. \quad (24)$$

After substitution into the WDW equation the Hamilton-Jacobi equation results and separating variables it is straightforward to obtain the following solution,

$$S = p_y y + p_+ \beta_+ + p_- \beta_- \pm \left[\frac{\sqrt{k^2 + \Lambda_0} e^{6x}}{3} - \frac{k \operatorname{ArcTanh}\left(\frac{\sqrt{k^2 + \Lambda_0} e^{6x}}{k}\right)}{3} - kx \right], \quad (25)$$

where $k^2 = p_y^2 + p_+^2 + p_-^2$. Once that we have solved the Hamilton-Jacobi equation it is possible to find the classical solution, we do not do that here because they were obtained by Banerjee et al.[17], solving the field equations.

3 Another potential

We consider now another potential function for which it is possible to obtain exact solutions to WDW equation. In Eq.(7) we change the time τ in the following way $d\tau = e^{-x}d\sigma$, the action becomes,

$$A = 6 \int d\sigma \left[e^{4x} \{ y'^2 - x'^2 + \beta_+'^2 + \beta_-'^2 \} - e^{2x} \Lambda(y) \right], \quad (26)$$

here the prime means derivative with respect to σ . We introduce a change of variables and an explicit potential,

$$\eta = e^{2x} \text{Cosh}(2y), \quad \xi = e^{2x} \text{Sinh}(2y), \quad \Lambda(y) = \Lambda_1 \text{Cosh}(2y) + \Lambda_2 \text{Sinh}(2y). \quad (27)$$

In the new time and variables, the action, Lagrangean, Hamiltonian can be calculated with the following results,

$$A = 6 \int d\sigma \left[\frac{1}{4}(\xi'^2 - \eta'^2) + (\eta^2 - \xi^2)(\beta_+'^2 + \beta_-'^2) - \Lambda_1 \eta - \Lambda_2 \xi \right], \quad (28)$$

$$L = 6 \left[\frac{1}{4}(\xi'^2 - \eta'^2) + (\eta^2 - \xi^2)(\beta_+'^2 + \beta_-'^2) - \Lambda_1 \eta - \Lambda_2 \xi \right], \quad (29)$$

$$H = \frac{1}{24} [\pi_\xi^2 - \pi_\eta^2 + (\eta^2 - \xi^2)(\pi_+^2 + \pi_-^2) + 24\Lambda_1 \eta + 24\Lambda_2 \xi]. \quad (30)$$

The WDW equations in this case is

$$\left[\partial_\xi^2 - \partial_\eta^2 + (\eta^2 - \xi^2)(\partial_+^2 + \partial_-^2) - 24\Lambda_1 \eta - 24\Lambda_2 \xi \right] \Psi(\eta, \xi, \beta_\pm) = 0. \quad (31)$$

Assuming separation of variables,

$$\Psi(\eta, \xi, \beta_\pm) = E(\eta) X(\xi) P(\beta_+) M(\beta_-), \quad (32)$$

the corresponding equations are

$$E'' + [k_\perp^2 \eta^2 + 24\Lambda_1 \eta + k^2] E = 0, \quad (33)$$

$$X'' + [k_{\perp}^2 \xi^2 - 24\Lambda_2 \xi + k^2]X = 0, \quad (34)$$

$$P'' + k_+^2 P = 0, \quad (35)$$

$$M'' + k_-^2 M = 0, \quad (36)$$

where k_+, k_- and k are arbitrary separation constants and $k_{\perp}^2 = k_+^2 + k_-^2$. The solutions are

$$\begin{aligned} E &= c_1 {}_1F_1(a_1, 1/2, z_1) + c_2 z_1^{1/2} {}_1F_1(a_1 + 1/2, 3/2, z_1), \\ a_1 &= \frac{1}{4}[-k^2 + (\frac{12\Lambda_1}{k_{\perp}})^2 \sqrt{-k_{\perp}^2} + 1], \quad z_1 = \sqrt{-k_{\perp}^2}(\eta + \frac{12\Lambda_1}{k_{\perp}^2})^2, \end{aligned} \quad (37)$$

$$\begin{aligned} X &= c_3 {}_1F_1(a_2, 1/2, z_2) + c_4 z_2^{1/2} {}_1F_1(a_2 + 1/2, 3/2, z_2), \\ a_2 &= \frac{1}{4}[-k^2 + (\frac{12\Lambda_2}{k_{\perp}})^2 \sqrt{-k_{\perp}^2} + 1], \quad z_2 = \sqrt{-k_{\perp}^2}(\xi + \frac{12\Lambda_2}{k_{\perp}^2})^2, \end{aligned} \quad (38)$$

$$P = c_5 e^{ik_+\beta_+} + c_6 e^{-ik_+\beta_+}, \quad (39)$$

$$M = c_7 e^{ik_-\beta_-} + c_8 e^{-ik_-\beta_-}, \quad (40)$$

where ${}_1F_1$ is the confluent hypergeometric function. More solutions can be obtained by superposition.

3.1 WKB solution

Again for the new potential we want to find the WKB wavefunction,

$$\Psi(\eta, \xi, \beta_{\pm}) = e^{i[V(\eta) + W(\xi) + (p_+\beta_+) + (p_-\beta_-)]}. \quad (41)$$

After substitution into the WDW equation we have

$$V_{\eta}^2 - p_{\perp}^2 \eta^2 - 24\Lambda_1 \eta - p^2 = 0, \quad (42)$$

$$W_\xi^2 - p_\perp^2 \xi^2 + 24\Lambda_2 \xi - p^2 = 0, \quad (43)$$

where $p_\perp^2 = p_+^2 + p_-^2$. The solutions are

$$V = \left(\frac{6\Lambda_1}{p_\perp^2} + \frac{\eta}{2} \right) \sqrt{p^2 + 24\Lambda_1 \eta + p_\perp^2 \eta^2} + \frac{(-144\Lambda_1^2 + p^2 p_\perp^2) \log\left(\frac{2(12\Lambda_1 + p_\perp^2 \eta)}{p_\perp} + 2\sqrt{p^2 + 24\Lambda_1 \eta + p_\perp^2 \eta^2}\right)}{2p_\perp^3}, \quad (44)$$

$$W = \left(-\frac{6\Lambda_2}{p_\perp^2} + \frac{\eta}{2} \right) \sqrt{p^2 - 24\Lambda_2 \eta + p_\perp^2 \eta^2} + \frac{(-144\Lambda_2^2 + p^2 p_\perp^2) \log\left(\frac{2(-12\Lambda_2 + p_\perp^2 \eta)}{p_\perp} + 2\sqrt{p^2 - 24\Lambda_2 \eta + p_\perp^2 \eta^2}\right)}{2p_\perp^3}. \quad (45)$$

4 Final remarks

In this work we have obtained exact solutions for the WDW equation using the general scalar tensor theory of gravitation with arbitrary coupling function $\omega(\Phi)$ and two specific selfinteracting potentials $\lambda(\Phi)$. The WKB wavefunctions were obtained for both cases. The solutions here obtained are for arbitrary large anisotropies. We hope that these solutions should prove to be useful in the study of the issues of quantum cosmology. In the past quantum cosmology using general relativity in homogeneous spacetimes with small anisotropies and a scalar field were considered by Lukash and Schmidt [21], and Amsterdamski [22]. Lidsey [23] has considered the wavefunctions in a highly anisotropic cosmologies with a massless minimally coupled scalar field. More recently Bachmann and Schmidt [24] have considered the case of arbitrary anisotropies in Bianchi I quantum cosmology.

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